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Thermodynamics of Black Hole in (N+3)-dimensions from Euclidean N-brane Theory

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ABSTRACT

In this article we consider an N-brane description of an (N+3)-dimensional black hole horizon. First of all, we start by reviewing a previous work where a string theory is used as describing the dynamics of the event horizon of a four dimensional black hole. Then we consider a particle model defined on one dimensional Euclidean line in a three dimensional black hole as a target spacetime metric. By solving the field equations we find a "world line instanton" which connects the past event horizon with the future one. This solution gives us the exact value of the Hawking temperature and to leading order the Bekenstein-Hawking formula of black hole entropy. We also show that this formalism is extensible to an arbitrary spacetime dimension. Finally we make a comment of one-loop quantum correction to the black hole entropy .

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The Bekenstein-Hawking formula of the black hole entropy, $S = \frac{1}{4}\frac{kc^3}{G\hbar}A_H$ [1,2] is not only so beautiful but also very mysterious for us at present. This formula contains the four fundamental constants of physics, those are, the Boltzman constant k, the Newton one G, the Planck one $\hbar$ and the light velocity c so that it suggests a deep triangle relation among thermodynamics, general relativity and quantum mechanics. Moreover, this formula relates the entropy of a black hole to the area of a event horizon, therefore also implies a connection to geometry. Thus, although the above formula was originally derived in terms of the semi-classical approach, many theoretical physicists never doubt its validity up to some quantum corrections even when we have a quantum theory of gravitation in the future.

However, the underlying physical basis by which $\frac{1}{4}\frac{kc^3}{G\hbar}A_H$ arises as the black hole entropy remains unclear. It is tempted to regard this black entropy as the logarithm of the number of microscopic states compatible with the observed macroscopic state from the viewpoint of the ordinary statistical physics. Then, a crux of an understanding is what those microscopic states are.

It might be true that the underlying law explaining the Bekenstein-Hawking formula might presumably not be fully understood until we construct a theory of quantum gravity. But there certainly exists an opposite attitude toward it. Namely this mystery might give us a clue of constructing a theory of quantum gravity. At this point one expects that a quantum black hole plays a similar role as the hydrogen atom at the advent of the quantum mechanics.

In a previous paper[3], we have considered a stringy description of a black hole horizon in four spacetime dimensions where we have described the event horizon of a black hole in terms of the world sheet swept by a string in the Rindler background. It was shown that a nonlinear sigma model action leads to both the Hawking temperature of a black hole and the well-known Bekenstein-Hawking formula of the black hole entropy within the lowest order of approximation. Furthermore we have derived a covariant operator algebra on the event horizon which is a natural

generalization to the 'tHooft one[4,5].

Thus it is natural to ask whether this stringy approach to the black hole thermodynamics can be extended to an arbitrary spacetime dimension or is a peculiar thing only in four dimensions. We will see later that we can in fact construct a more general formalism where the event horizon of a black hole in (N+3)-spacetime dimensions is described by a Euclidean N-brane.

We start with a brief review of the previous work on a string, i.e., 1-brane, approach to the four dimensional black hole[3]. As an effective action describing the dynamical properties of the black hole horizon, let us consider the Polyakov action of a bosonic string in a curved target spacetime which is given by

$$S_{(1)} = -\frac{T}{2} \int d^2\sigma \sqrt{h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}(X), \tag{1}$$

where T is a string tension having dimensions of mass squared, $h_{\alpha\beta}(\tau, \sigma)$ denotes the two dimensional world sheet metric having a Euclidean signature, and $h = det h_{\alpha\beta}$. $X^\mu(\tau, \sigma)$ maps the string into four dimensional spacetime, and then $g_{\mu\nu}(X)$ can be identified as the background spacetime metric in which the string is propagating. Note that α, β take values 0, 1 and μ, ν do values 0, 1, 2, 3.

The classical field equations from the action (1) become

$$\begin{aligned} 0 = T_{\alpha\beta} &= -\frac{2}{T} \frac{1}{\sqrt{h}} \frac{\delta S_{(1)}}{\delta h^{\alpha\beta}}, \\ &= \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}(X) - \frac{1}{2} h_{\alpha\beta} h^{\rho\sigma} \partial_\rho X^\mu \partial_\sigma X^\nu g_{\mu\nu}(X), \end{aligned} \tag{2}$$

$$0 = \partial_\alpha (\sqrt{h} h^{\alpha\beta} g_{\mu\nu} \partial_\beta X^\nu) - \frac{1}{2} \sqrt{h} h^{\alpha\beta} \partial_\alpha X^\rho \partial_\beta X^\sigma \partial_\mu g_{\rho\sigma}. \tag{3}$$

Here let us consider the case where the background spacetime metric $g_{\mu\nu}(X)$ takes

a form of the Euclidean Rindler metric

$$ds^2 = g_{\mu\nu} dX^\mu dX^\nu = +g^2 z^2 dt^2 + dx^2 + dy^2 + dz^2, \tag{4}$$

where g is given by $g = \frac{1}{4M}$. It is well-known that the Rindler metric can be obtained in the large mass limit from the Schwarzschild black hole metric and provides us with a nice playground for examining the Hawking radiation in a simple metric form[6,7]. Here it is important to notice that we have performed the Wick rotation with respect to the time component since now we would like to discuss the thermodynamic properties of the Rindler spacetime when we assume that the dynamics of the event horizon is controlled by a Euclidean string.

Now one can easily solve Eq.(2) as follows:

$$h_{\alpha\beta} = G(\tau, \sigma) \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}(X), \tag{5}$$

where $G(\tau, \sigma)$ denotes the Liouville mode. Next we shall fix the gauge symmetries which are the two dimensional diffeomorphisms and the Weyl rescaling by

$$x(\tau, \sigma) = \tau, \ y(\tau, \sigma) = \sigma, \ G(\tau, \sigma) = 1. \tag{6}$$

At this stage, let us impose a cyclic symmetry

$$z(\tau, \sigma) = z(\tau), \ t(\tau, \sigma) = t(\tau). \tag{7}$$

Some people might be afraid that this additional assumption kills various physically interesting states by which one cannot derive the entropy formula, but we will see that such a situation never occurs at least at the lowest order of approximation. Incidentally, we will not impose the same kind of symmetry ansatz in the case of

three dimensional black hole. From Eqs.(5), (6) and (7), the world sheet metric takes the form

$$h_{\alpha\beta} = \begin{pmatrix} g^2 z^2 \dot{t}^2 + \dot{z}^2 + 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{8}$$

where the dot denotes a derivative with respect to τ. And the remaining field equations (3) become

$$\partial_\tau (\frac{z^2 \dot{t}}{\sqrt{h}}) = 0, \tag{9}$$

$$\partial_\tau h = 0, \tag{10}$$

$$\partial_\tau (\frac{\dot{z}}{\sqrt{h}}) - \frac{1}{\sqrt{h}} g^2 z \dot{t}^2 = 0, \tag{11}$$

where

$$h = g^2 z^2 \dot{t}^2 + \dot{z}^2 + 1. \tag{12}$$

Now it is straightforward to solve the above field equations. We have two kinds of solutions. One is a trivial one given by

$$z = \dot{z} = \ddot{z} = 0, t(\tau) = arbitrary. \tag{13}$$

The other is the solution of "world sheet instanton" described by

$$\begin{aligned} z(\tau) &= \sqrt{c_2 (\tau - \tau_0)^2 + \frac{g^2 c_1^2}{c_2}}, \\ t - t_0 &= \frac{1}{g} \tan^{-1} \frac{c_2}{g c_1} (\tau - \tau_0), \end{aligned} \tag{14}$$

where c_1, c_2, τ_0, and t_0 are the integration parameters, in other words, "moduli parameters". In order to understand the physical meaning of this solution more

vividly, it is convenient to eliminate the variable τ and express z in terms of the time variable t. Then from Eq.(14) we obtain

$$z(t_E) \ = \ \frac{gc_1}{\sqrt{c_2}} \frac{1}{\cos g(t_E - t_{E0})}, \tag{15}$$

where we inserted the suffix E on t in order to indicate the Euclidean time clearly. Furthermore after Wick-rerotating, we have in the real Lorentzian time t_L

$$z(t_L) \ = \ \frac{gc_1}{\sqrt{c_2}} \frac{1}{\cosh g(t_L - t_{L0})}. \tag{16}$$

Here note that the Rindler coordinate (z, t) is related to the Minkowski coodinate (Z, T) by the following transformation:

$$Z = z \cosh gt, \ T = t \sinh gt, \tag{17}$$

thus the above solution (16) corresponds to an instanton connecting the past event horizon with the future one with a definite constant value $Z = \frac{gc_1}{\sqrt{c_2}}$.

Now let us examine the thermodynamic properties of the "world sheet instanton". It is remarkable that the solution has a periodicity with respect to the Euclidean time component, $\beta = \frac{2\pi}{g}$ whose inverse gives us nothing but the Hawking temperature $T_H = \frac{1}{\beta} = \frac{g}{2\pi} = \frac{1}{8\pi M}$ of the Rindler spacetime[6]. Next let us calculate the black hole entropy to the leading order of approximaion by a method developed by Gibbons and Hawking[8]. The result is

$$S = \sqrt{c_2 + 1} \ T \ A_H, \tag{18}$$

where $A_H = \int dxdy$ which corresponds to the area of the black hole horizon. At

this stage, by selecting the string tension

$$T = \frac{1}{4\sqrt{c_2 + 1}G}, \tag{19}$$

we arrive at the famous Bekenstein-Hawking entropy formula[1,2]

$$S = \frac{1}{4G} A_H. \tag{20}$$

Let us apply the formalism mentioned so far for a three dimensional black hole model[9] whose line element is given by

$$ds^2 = -(-M + \frac{r^2}{l^2})dt^2 + (-M + \frac{r^2}{l^2})^{-1} dr^2 + r^2 d\phi^2, \tag{21}$$

where l^2 and the cosmological constant Λ have a relation like $\Lambda = -\frac{1}{l^2}$. From Eq.(21), one can have the Rindler metric in the limit of a small cosmological constant after an appropriate change of variables. In fact, defining

$$x = l\sqrt{M}\phi, \ z = \frac{l}{\sqrt{M}}\sqrt{-M + \frac{r^2}{l^2}}, \tag{22}$$

and identifying $g = \frac{\sqrt{M}}{l}$, one has the following Rindler metric

$$ds^2 = -g^2 z^2 dt^2 + dx^2 + dz^2. \tag{23}$$

Note that the black hole metric Eq.(21) and the Rindler metric Eq.(23) differ only by terms $O((\frac{z}{l})^2)$.

In case of the three dimensional Rindler spacetime, we need to consider a particle (0-brane) model defined on one dimensional Euclidean line with the Euclidean Rindler metric as a spacetime metric. Thus we start by an action

$$S_{(0)} = -\frac{T}{2} \int d\tau (\ \frac{1}{e} \dot{X}^\mu \dot{X}^\nu g_{\mu\nu}(X) + e \ ), \tag{24}$$

where e is the einbein. Note that now T must have dimensionality of mass to leave a dimensionless action. Of course, in this case, μ, ν take values 0, 1, 2.

From Eq.(24), one has the field equations through the variation of the einbein and X^μ

$$0 = \dot{X}^\mu \dot{X}^\nu g_{\mu\nu}(X) - e^2, \tag{25}$$

$$0 = \partial_\tau (\ \frac{1}{e} \ g_{\mu\nu} \dot{X}^\nu \ ) - \frac{1}{2e} \dot{X}^\rho \dot{X}^\sigma \partial_\mu g_{\rho\sigma}. \tag{26}$$

Now we shall fix the gauge symmetry which is one dimensional diffeomorphism by

$$x(\tau) = \tau. \tag{27}$$

From Eqs.(25), (26) and (27), it is easy to show that the field equations can be rewritten into a perfectly equivalent form to Eqs.(9)-(12) by defining $h = e^2$.

In a completely analogous way to the string case, we can obtain the "world line instanton" solution Eq.(15). This solution gives us the Hawking temperature $T_H = \frac{1}{\beta} = \frac{g}{2\pi} = \frac{\sqrt{M}}{2\pi l}$ of the three dimensional black hole[9]. Now the entropy is given by

$$S = \sqrt{c_2 + 1} \ T \ L_H, \tag{28}$$

where $L_H = \int dx$ denotes the length of the black hole horizon. Since both the particle coupling constant T and the inverse of the Newton constant G in three

dimensions have a dimension of mass, one can set up the relation (19) and then reach the Bekenstein-Hawking entropy formula (20) where A_H is now replaced with L_H.

Let us generalize the present formalism to the black hole in (N+3)-dimensions where we need to consider the Euclidean N-brane. We will assume $N \geq 2$ without generality. Our starting action as an effective action describing the dynamics of the event horizon is given by

$$S_{(N)} = -\frac{T}{2} \int d^{N+1}\sigma \sqrt{h} (\ h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}(X) + 1 - N \ ), \tag{29}$$

where T is an N-brane tension having dimensionality $(mass)^{N+1}$. Now α, β take values 0, 1,..., N and μ, ν do values 0, 1,..., N+2. The variation of the action (29) gives us the field equations

$$\begin{aligned} 0 = T_{\alpha\beta} &= -\frac{2}{T} \frac{1}{\sqrt{h}} \frac{\delta S_{(N)}}{\delta h^{\alpha\beta}}, \\ &= \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}(X) - \frac{1}{2} h_{\alpha\beta} (\ h^{\rho\sigma} \partial_\rho X^\mu \partial_\sigma X^\nu g_{\mu\nu}(X) + 1 - N \ ), \end{aligned} \tag{30}$$

and Eq.(3). The Schwarzschild black hole metric in (N+3)-spacetime dimensions takes a form in a Euclidean signature

$$ds^2 = +(1 - \frac{2M}{r}) dt^2 + (1 - \frac{2M}{r})^{-1} dr^2 + r^2 d\Omega^2_{N+1}, \tag{31}$$

which in the large mass limit can be written to the Rindler metric

$$ds^2 = +g^2 z^2 dt^2 + dz^2 + \sum_{i=1}^{N+1} dx_i^2, \tag{32}$$

with $g = \frac{1}{4M}$.

As before, one can solve Eq.(30) as follows:

$$h_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}(X). \tag{33}$$

The gauge symmetries, which are (N+1)-dimensional diffeomorphisms, are fixed by the gauge conditions

$$\begin{aligned} x_1(\tau, \sigma_1, \cdots, \sigma_N) &= \tau, \\ x_2(\tau, \sigma_1, \cdots, \sigma_N) &= \sigma_1, \\ &\cdots\cdots \\ x_{N+1}(\tau, \sigma_1, \cdots, \sigma_N) &= \sigma_N. \end{aligned} \tag{34}$$

Furthermore we shall make a cyclic ansatz

$$\begin{aligned} z(\tau, \sigma_1, \cdots, \sigma_N) &= z(\tau), \\ t(\tau, \sigma_1, \cdots, \sigma_N) &= t(\tau). \end{aligned} \tag{35}$$

From Eqs.(33), (34) and (35), the world volume metric becomes

$$h_{\alpha\beta} = \begin{pmatrix} g^2 z^2 \dot{t}^2 + \dot{z}^2 + 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}. \tag{36}$$

As the remaining field equations we have the same equations as Eqs.(9)-(12) ,thus as an interesting classical solution “world volume instanton” Eq.(15). Of course this solution has a periodicity whose inverse is exactly equal to the Hawking temperature, and the entropy is given in terms of the Bekenstein-Hawking formula Eq.(20) where this time A_H denotes the volume of the event horizon given by $\int \prod_{i=1}^{N+1} dx_i$.

Finally let us make a comment on one-loop quantum correction to the black hole entropy. Recently there have appeared several articles where string theories have been utilized[10,11,12,13]. An idea behind these researches is that string theories might provide a finite quantum correction to the Bekenstein-Hawking entropy formula owing to a mild ultraviolet behavior at a short distance scale. However, from the viewpoint of the present work, just in four spacetime dimensions one has to use Euclidean string theories in order to understand the quantum mechanical meaning of the black hole entropy. We would like to consider this problem in the near future.

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